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Best Approximation of Bounded Functions by Continuous Functions

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0. INTRODUCTION

It is a classical result that every closed subalgebra of $C(X; \mathbb{R})$, X compact, is a proximinal subspace. Proofs of this result were published by Pelczynski and Semadeni, but the result itself was known to Mazur. Several generalizations and extensions of Mazur's theorem have appeared in the literature. For example, one may consider the problem of existence of Chebyshev centers for bounded subsets of $C(X; \mathbb{R})$, i.e., the problem of deciding when $C(X; \mathbb{R})$ admits centers. This was settled by Kadets and Zamyatin, for $X = [a, b]$, and by Garkavi, for X compact (see [13] and [12]).

The existence of relative Chebyshev centers (also called restricted centers) with respect to a closed subalgebra $A \subset C(X : \mathbb{R})$ was established in 1975 by Smith and Ward (see [24]), for any compact space X. An extension of this result to bounded functions, i.e., to closed subalgebras of $l_\infty(X; \mathbb{R})$ was obtained by Mach in 1979 (see [18]).

Another line of generalizations of Mazur's theorem consists of the consideration of vector-valued functions. The problem of the existence of Chebyshev centers for bounded subsets of $C_b(X; E)$, the space of continuous and bounded E-valued functions, was solved by Ward in 1974 (see [25]) in the following two cases: (a) E is a finite-dimensional strictly convex normed space and X is paracompact; (b) E is a Hilbert space and X is normal. Amir in 1978 (see [1]) generalized both results by proving that $C_b(X; E)$ admits centers when E is a uniformly convex Banach space and X is any topological space.

When $W \subset C_b(X; E)$ is a closed vector subspace one asks whether it is proximinal or, more generally, whether any bounded subset of $C_b(X; E)$ has a relative Chebyshev center with respect to W. Along this line we have the study of proximinality of Grothendieck subspaces (in particular Stone–

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Weierstrass and Kakutani subspaces) $W \subset C_0(X; E)$ made by Blatter (see [4]). For example, when E is a real Lindenstrauss space, any Stone–Weierstrass subspace $W \subset C_0(X; E)$ is proximinal [4, Corollary 3.19]; see also Yost [26, Theorem 2.1]. The case in which E is a uniformly convex Banach space, and W is a Stone–Weierstrass subspace was studied by Olech (see [21]) who proved proximinality of such subspaces in $C(X; E)$, X compact. Lau (see [15]) extended this to $l_\infty(X; E)$. In 1979, Mach (see [18]) generalized Olech's result showing that any bounded subset $B \subset C(X; E)$, X compact and E uniformly convex, has a restricted center with respect to any Stone–Weierstrass subspace $W \subset C(X; E)$.

These results of the literature are particular cases of our theorems. We apply our results to establish existence of Chebyshev centers with respect to $\mathscr{K}(E; C_0(X))$, the space of compact linear operators from a uniformly smooth space E into $C_0(X)$, X locally compact, for bounded subsets B of $\mathscr{L}(E; C_0(X))$.

1. Definitions and Notations

Let $(E, \| \ \|)$ be a normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If B is a bounded set in E, we denote $r(x, B) = \sup\{\|x - b\|; b \in B\}$, where $x \in E$. For any nonempty subset M of E, we define the *relative Chebyshev radius of B* with respect to M to be

$$\operatorname{rad}(B, M) = \inf\{r(x, B); x \in M\}$$

and the set of *Chebyshev centers of B* in M to be

$$\operatorname{cent}(B, M) = \{x \in M; r(x, B) = \operatorname{rad}(B, M)\}.$$

Elements of $\operatorname{cent}(B, M)$ are also called *best simultaneous approximations* of B by elements of M. When B is a single point y,

$$\operatorname{rad}(B, M) = \operatorname{dist}(y, M)$$

and

$$\operatorname{cent}(B, M) = P_M(y).$$

We say that M has the *relative Chebyshev center property in E* if $\operatorname{cent}(B, M) \neq \varnothing$ for every bounded set B in E. When $M = E$ has this property, we say that E *admits Chebyshev centers*. Notice that if *M has the relative Chebyshev center property in E then M is proximinal in E*. Finally, for each $\varepsilon > 0$ define

$$\varphi_\varepsilon(u, v) = v \qquad \text{if} \quad \|u - v\| \leqslant \varepsilon$$

and

$$\varphi_\varepsilon(u, v) = (1 - \varepsilon \|u - v\|^{-1})\, u + \varepsilon \|u - v\|^{-1} v,$$

otherwise, for u and v in E. The mapping

$$(u, v) \in E \times E \mapsto \varphi_\varepsilon(u, v) \in E$$

is continuous and satisfies

$$\|\varphi_\varepsilon(u, v) - u\| \leqslant \varepsilon$$

for all $u, v \in E$. We say that $(E, \|\ \|)$ has *property* (P) if for every $r > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\bar{B}(u, r + \delta) \cap \bar{B}(v, r + \theta) \subset \bar{B}(\varphi_\varepsilon(u, v), r + \theta)$$

for all $0 < \theta < \delta$ and $u, v \in E$, where $\bar{B}(y, s)$ denotes the closed ball with center in y and radius s. The most important class of spaces having property (P) is that of all *uniformly convex Banach spaces* (see Mach [17]).

2. Spaces of Bounded Mappings

Let X be a nonempty set and $(E, \|\ \|)$ be a Banach space; $l_\infty(X; E)$ denotes the space of all bounded E-valued functions f on X with the sup–norm $\|f\| = \sup\{\|f(x)\|; x \in X\}$. When $X = \mathbb{N}$, we write $l_\infty(\mathbb{N}; E) = l_\infty(E)$ and $l_\infty(\mathbb{N}; \mathbb{K}) = l_\infty$.

Definition 2.1. Let $\varepsilon > 0$ be given. For $f, g \in l_\infty(X; E)$ we define a function $h_\varepsilon(f, g)$ by

$$h_\varepsilon(f, g)(x) = \varphi_\varepsilon(f(x), g(x))$$

for all $x \in X$, where φ_ε is the mapping that we defined in the introduction.

It follows that $h_\varepsilon(f, g) \in l_\infty(X; E)$ and $\|h_\varepsilon(f, g) - f\| \leqslant \varepsilon$, for all $f, g \in l_\infty(X; E)$.

Theorem 2.2. *Let E be a Banach space with property (P). Let $W \subset l_\infty(X; E)$ be a closed subset such that $h_\varepsilon(f, g) \in W$ for all $f, g \in W$ and $\varepsilon > 0$. Then W has the relative Chebyshev center property in $l_\infty(X; E)$. In particular, $l_\infty(X; E)$ admits Chebyshev centers.*

Proof. Let $r > 0$ and $\varepsilon > 0$ be given. By the hypothesis, h_ε maps $W \times W$

into W. Take $\delta > 0$ as in property (P) and let $f, g \in l_\infty(X; E)$ and $0 < \theta < \delta$ be given. If

$$w \in \bar{B}(f, r+\delta) \cap \bar{B}(g, r+\theta),$$

then by property (P),

$$w(x) \in \bar{B}(\varphi_\varepsilon(f(x), g(x)), r+\theta)$$

for every $x \in X$. Hence,

$$w \in \bar{B}(h_\varepsilon(f, g), r+\theta).$$

By Theorem 2 of Mach [17], W has the relative Chebyshev center property in $l_\infty(X; E)$.

COROLLARY 2.3. *Let E be a Banach space with property (P). Every closed convex $l_\infty(X; [0, 1])$-submodule W of $l_\infty(X; E)$ has the relative Chebyshev center property in $l_\infty(X; E)$.*

Proof. Let $f, g \in W$ and $\varepsilon > 0$ be given. If β_ε is the real-valued function defined by $\beta_\varepsilon(t) = 1$ when $|t| \leqslant \varepsilon$ and $\beta_\varepsilon(t) = \varepsilon t^{-1}$ otherwise, then

$$\varphi_\varepsilon(f(x), g(x)) = f(x) + \beta_\varepsilon(\|f(x) - g(x)\|)[g(x) - f(x)]$$

for every $x \in X$. Defining $\psi_{\varepsilon, f, g} \in l_\infty(X; [0, 1])$ by

$$\psi_{\varepsilon, f, g}(x) = \beta_\varepsilon(\|f(x) - g(x)\|)$$

for all $x \in X$, we have

$$h_\varepsilon(f, g) = (1 - \psi_{\varepsilon, f, g}) f + \psi_{\varepsilon, f, g} \cdot g,$$

and thus $h_\varepsilon(f, g) \in W$.

Let $E = \mathbb{R}$ normed by its usual absolute value. For $\varepsilon > 0$ and $f, g \in l_\infty(X; E)$, we can write $h_\varepsilon(f, g)$ in the form

$$h_\varepsilon(f, g) = \sup(\inf(g, f + \varepsilon), f - \varepsilon).$$

This and Theorem 2.2 give

PROPOSITION 2.4. *Let L be a closed sublattice of $l_\infty(X; \mathbb{R})$ such that $f \pm \varepsilon$ belong to L for all $f \in L$ and $\varepsilon > 0$. Then L has the relative Chebyshev center property in $l_\infty(X; \mathbb{R})$.*

EXAMPLE 2.5. Let $(X, \leqslant)$ be a preordered set and J be the set of all nondecreasing elements of $l_\infty(X; \mathbb{R})$. It is easy to see that J is a closed sublattice of $l_\infty(X; \mathbb{R})$ and $f \pm \varepsilon$ belong to J, for all $f \in J$ and $\varepsilon > 0$. Thus J has the relative Chebyshev center property in $l_\infty(X; \mathbb{R})$.

For $a, b \in l_\infty(X; \mathbb{R})$, let $[a, b]$ denote the order interval $\{h \in l_\infty(X; \mathbb{R}): a(x) \leqslant h(x) \leqslant b(x),\ x \in X\}$. It is easy to see that $h_\varepsilon(f, g) \in [a, b]$ for all $f, g \in [a, b]$ and $\varepsilon > 0$. Thus, by Theorem 2.2 we have

PROPOSITION 2.6. *Each order interval $[a, b] \subset l_\infty(X; \mathbb{R})$ has the relative Chebyshev center property in $l_\infty(X; \mathbb{R})$.*

Franchetti and Cheney [11, Lemma 3.5] proved the proximinality of any order interval $[a, b] \subset E$ when E is a Banach lattice. Hence, our Proposition 2.6 extends Lemma 3.5 of [11] when E is the Banach lattice $l_\infty(X; \mathbb{R})$.

3. SPACES OF CONTINUOUS BOUNDED MAPPINGS

Let X be a topological space, $(E, \|\ \|)$ be a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and $C(X; E)$ be the space of all continuous E-valued functions on X. Let

$$C_b(X; E) = C(X; E) \cap l_\infty(X; E),$$

with the sup–norm induced by $l_\infty(X; E)$.

THEOREM 3.1. *Suppose that E has property (P). Then every closed $C_b(X; \mathbb{K})$-submodule of $C_b(X; E)$ has the relative Chebyshev center property in $l_\infty(X; E)$.*

Proof. Similar to Corollary 2.3.

COROLLARY 3.2. *If E has property (P) then $C_b(X; E)$ has the relative Chebyshev center property in $l_\infty(X; E)$. In particular, $C_b(X; E)$ admits Chebyshev centers.*

Remark. Kadets–Zamyatin [13] proved that $C_b(X; E)$ admits Chebyshev centers when $X = [a, b] \subset \mathbb{R}$ and $E = \mathbb{R}$. Ward [25] generalized this result in the following two cases: (a) E is a finite dimensional strictly convex normed space and X is paracompact; (b) E is a Hilbert space and X is normal. Amir [1] generalized all these results proving that $C_b(X; E)$ admits Chebyshev centers when E is a uniformly convex Banach space. Notice that in all three cases E has property (P).

COROLLARY 3.3. *Suppose that E has property (P) and $Z \subset X$ is a closed subset. Then $W = \{f \in C_b(X; E);\ f(x) = 0,\ x \in Z\}$ has the relative Chebyshev center property in $l_\infty(X; E)$.*

Proof. Clearly, W is closed $C_b(X; \mathbb{K})$-submodule of $C_b(X; E)$.

PROPOSITION 3.4. *Let Y be a topological space and suppose that E has property (P). Let $M \subset C_b(Y; E)$ be a closed $C_b(Y; \mathbb{K})$-submodule and π a surjection from a set Z onto Y. Then $\pi^0(M)$ has the relative Chebyshev center property in $l_\infty(Z; E)$, where $\pi^0(f) = f \circ \pi$ for all $f \in M$.*

Proof. Define a topology on Z as follows: $A \subseteq Z$ is open if and only if $A = \pi^{-1}(B)$ with B open in Y. Thus $C_b(Y; E)$ is isometrically isomorphic to $C_b(Z; E)$ under π^0, and $\pi^0(M)$ is a closed $C_b(Z; \mathbb{K})$-submodule of $C_b(Z; E)$. The result follows from Theorem 3.1.

Remark. When E is uniformly convex and Y, M, and π are as in Proposition 3.4 Lau [15, Theorem 4.3] proved that $\pi^0(M)$ is proximinal.

THEOREM 3.5. *Let E be a Banach space with property (P). Every Stone–Weierstrass subspace W of $C_b(X; E)$ has the relative Chebyshev center property in $l_\infty(X; E)$.*

Proof. By definition, a Stone–Weierstrass subspace W has the form

$$W = \{g \circ \pi; g \in C_b(Y; E)\},$$

where Y is a topological space and π is a closed continuous surjection of X onto Y. Thus $W = \pi^0(C_b(Y; E))$ and the result follows from Proposition 3.4.

COROLLARY 3.6. *Let X be a compact space and suppose that E has property (P). Then every Stone–Weierstrass subspace of $C(X; E)$ has the relative Chebyshev center property in $l_\infty(X; E)$.*

When E is uniformly convex and X is compact, Olech [21] proved the proximinality of the Stone–Weierstrass subspaces in $C(X; E)$. Mach [18] generalized Olech's result by proving that such subspaces have the Chebyshev center property in $C(X; E)$, under the hypothesis that E is uniformly convex and X is compact.

PROPOSITION 3.7. *Let E be a Banach space with property (P). Every closed subset $W \subset C_b(X; E)$ which is a convex $C(X; [0, 1])$-submodule has the relative Chebyshev center property in $l_\infty(X; E)$ and, a fortiori, in $C_b(X; E)$.*

Proof. Write $h_\varepsilon(f, g)$ as in Corollary 2.3 and notice that the mappings $\psi_{\varepsilon, f, g}$ belong to $C(X; [0, 1])$.

EXAMPLES 3.8. (a) If M is a closed convex subset of a Banach space E, let

$$C_b(X; M) = \{f \in C_b(X; E); f(X) \subset M\}.$$

It is easy to see that $C_b(X; M)$ is a closed convex $C(X; [0, 1])$-submodule, contained in $l_\infty(X; E)$.

(b) The space $C^*(X; E) = \{f \in C_b(X; E); f(X)$ is compact in $E\}$ is a closed convex $C(X; [0, 1])$-submodule contained in $l_\infty(X; E)$.

THEOREM 3.9. *Let X be a compact space and E a Banach space with property (P). Every closed and self-adjoint polynomial algebra $W \subset C(X; E)$ has the relative Chebyshev center property in $l_\infty(X; E)$ and, a fortiori, in $C(X; E)$.*

Proof. For every $f, g \in W$ and $\varepsilon > 0$, one has $h_\varepsilon(f, g) \in W$, by Theorem 4.17 of Prolla [22].

COROLLARY 3.10. *Let X be a nonempty set. Every closed and self-adjoint subalgebra A of $l_\infty(X; \mathbb{K})$ has the relative Chebyshev center property in $l_\infty(X; \mathbb{K})$. If X is a topological space then every closed and self-adjoint subalgebra A of $C_b(X; \mathbb{K})$ has the relative Chebyshev center property in $l_\infty(X; \mathbb{K})$ and, a fortiori, in $C_b(X; \mathbb{K})$.*

Proof. Denote by X_d the topological space obtained endowing X with the discrete topology. Then $l_\infty(X; \mathbb{K}) = C_b(X_d, \mathbb{K})$. Since X_d is completely regular and Hausdorff, $C_d(X_d; \mathbb{K})$ is isometrically isomorphic as a C^*-algebra to $C(\beta X_d; \mathbb{K})$. Since the definitions of subalgebra and polynomial subalgebra coincide in $C(\beta X_d; \mathbb{K})$, the result follows from Theorem 3.9. When X is a topological space, every closed subalgebra of $C_b(X; \mathbb{K})$ is closed in $l_\infty(X; \mathbb{K})$, and the result follows from the first part.

EXAMPLE 3.11. Let X be a nonempty set and let L be a sublattice of the power set 2^X (the lattice operations being $\cup$ and $\cap$), containing ϕ and X. If $f: X \to \mathbb{R}$, one says that f is L-continuous if $f(Y) \in L$ for every closed subset $Y \subset \mathbb{R}$. We will denote by $C(L)$ the vector space of all real-valued L-continuous functions, and by $C_b(L)$ the vector subspace of $C(L)$ of all bounded real-valued L-continuous functions. When L is a *delta lattice* (i.e., closed under countable intersections) then $C_b(L)$ is a Banach algebra under the sup–norm, i.e., $C_b(L)$ is a closed subalgebra of $l_\infty(X; \mathbb{R})$. By Corollary 3.10, $C_b(L)$ has the relative Chebyshev center property in $l_\infty(X; \mathbb{R})$, for any delta lattice L. For the importance of $C_b(L)$ and $C(L)$, see Bachman and Sultan [3].

Smith and Ward [24] proved that every closed subalgebra of $C(X; \mathbb{R})$ has the relative Chebyshev center property in $C(X; \mathbb{R})$ when X is compact. Yost proved that such subalgebras have the $1\frac{1}{2}$-ball property and, therefore, are proximinal in $C(X; \mathbb{R})$ (see [26, Lemma 1.1]).

PROPOSITION 3.12. *Let u, v be two functions in $l_\infty(X; \mathbb{R})$ such that $u \leqslant v$. If the set $I = \{f \in C_b(X; \mathbb{R}); u \leqslant f \leqslant v\}$ is nonempty, then it has the relative Chebyshev center property in $l_\infty(X; \mathbb{R})$.*

Proof. For $\varepsilon > 0$ and $f, g \in I$, it is easy to see that the mapping $h_\varepsilon(f, g)$ belongs to I, since $h_\varepsilon(f, g)$ is continuous when f and g are.

Remark. Given any two real-valued functions u and v on a topological space X the problem arises of inserting a continuous function f between u and v, i.e., to find a continuous mapping f such that $u \leqslant f \leqslant v$. More generally, let P_1 and P_2 be two classes of real-valued functions on a topological space X such that they contain the constant functions and $P_i + C(X; \mathbb{R}) \subset P_i$ $(i = 1, 2)$. A space X has the *weak insertion property for* (P_1, P_2) if and only if for any pair of functions (f_1, f_2) with $f_1 \leqslant f_2, f_i \in P_i$ $(i = 1, 2)$ there exists a continuous function f on X such that $f_1 \leqslant f \leqslant f_2$ (Lane [14, p. 181]).

In the following the abbreviations lsc and usc are used for lower semicontinuous and upper semicontinuous. The well-known Tong–Katetov theorem states that a space X has *the weak insertion property for* (usc, lsc) if and only if *X is normal.* If one reverses the roles of upper and lower semicontinuity, the following is true (see Stone [23], Lane [14]): a space X has the *weak insertion property for* (lsc, usc) if and only if X is *extremally disconnected* (i.e., the closure of each open set is open).

If $\mathscr{C}$ is a collection of subsets of X, a function $f: X \to \mathbb{R}$ is called *$\mathscr{C}$-lower* (resp. *$\mathscr{C}$-upper*) *semicontinuous* if, for any $r \in \mathbb{R}$, the set $\{x \in X; f(x) > r\}$ (resp. the set $\{x \in X; f(x) \geqslant r\}$) belongs to $\mathscr{C}$. The abbreviations $\mathscr{C}$-lsc and $\mathscr{C}$-usc are used for $\mathscr{C}$-lower semicontinuous and $\mathscr{C}$-upper semicontinuous. Notice that $\mathscr{C}$-lsc reduces to lsc when $\mathscr{C}$ is the collection of all open subsets of X; and that $\mathscr{C}$-usc reduces to usc if $\mathscr{C}$ is the collection of all closed subsets of X.

Let $\mathscr{E}$ denote the class of all *zero sets* of X, that is,

$$\mathscr{E} = \{Z(f); f \in C(X; \mathbb{R})\},$$

where, for each $f \in C(X; \mathbb{R})$, $Z(f) = \{x \in X; f(x) = 0\}$. The following result is due to Stone (see Blatter and Seever [5] and Lane [14]): if X is *basically disconnected* (i.e., the closure of the complement of each zero set is open), then X has the *weak insertion property* for (lsc, $\mathscr{E}$-usc) and for ($\mathscr{E}^c$-lsc, usc).

Here $\mathscr{Z}^c$ denotes the class of all *cozero* sets, i.e., the complements of the zero sets.

There is a case in which it is possible to insert a continuous function with no restriction on X. Namely, the following is true (see [5, Proposition 6.1, p. 41]): any topological space X has the *weak insertion property for* ($\mathscr{Z}$-usc, $\mathscr{Z}^c$-lsc). Reversing the roles of usc and lsc one gets the following ([5, Proposition 6.3, p. 42]): if X is a *P-space*, i.e., every zero set is a cozero set, then X has the *weak insertion property for* ($\mathscr{Z}^c$-lsc, $\mathscr{Z}$-usc).

Let $f: X \to \mathbb{R}$ be given. The upper and lower semicontinuous regularizations of f are defined as

$$f^*(x) = \limsup_{y \to x} f(y),$$

$$f_*(x) = \liminf_{y \to x} f(y).$$

A mapping f is said to be *normal*-lsc if $f = (f^*)_*$; and *normal*-usc if $f = (f_*)^*$. The following results are due to Lane [14]. A space X has the *weak insertion property for* (normal usc, normal lsc) if and only if X *is mildly normal.* (A space is *mildly normal* in case disjoint regular closed subsets are separated by disjoint open sets. A subset is regular closed if it is equal to the closure of its interior). A space X has the *weak insertion property for* (normal-usc, lsc) (resp. (usc, normal-lsc)) if and only if X *is almost normal.* (A space is *almost normal* in case disjoint closed sets, at least one of which is regular closed, are separated by disjoint open sets.)

Combining these results with Proposition 3.12 one gets

THEOREM 3.13. *Let u and v be functions in $l_\infty(X; \mathbb{R})$ such that $u \leqslant v$. The set $I = \{f \in C_b(X; \mathbb{R});\ u \leqslant f \leqslant v\}$ has the relative Chebyshev center property in $l_\infty(X; \mathbb{R})$ in the following cases*:

(a) *X is any topological space, u is $\mathscr{Z}$*-usc, *and v is $\mathscr{Z}^c$*-lsc;

(b) *X is a P-space, u is $\mathscr{Z}^c$*-lsc, *and v is $\mathscr{Z}$*-usc;

(c) *X is normal, u is* usc, *and v is* lsc;

(d) *X is extremally disconnected, u is* lsc, *and v is* usc;

(e) *X is basically disconnected, u is* lsc (*resp.* $\mathscr{Z}^c$-lsc), *and v is* $\mathscr{Z}$-usc (*resp.* usc);

(f) *X is mildly normal, u is normal*-usc, *and v is normal*-lsc;

(g) *X is almost normal, u is normal*-usc (*resp.* usc), *and v is* lsc (*resp. normal*-lsc).

Remark. Part (c) of Theorem 3.13 generalizes Corollary 3.7 of Franchetti and Cheney [11].

In the remainder of this section X is a locally compact space and $C_0(X; E)$ is the subspace of all continuous E-valued functions on X vanishing at infinity. When $X = \mathbb{N}$ we write $c_0(E) = C_0(\mathbb{N}; E)$ and $c_0 = C_0(\mathbb{N}; \mathbb{K})$.

THEOREM 3.14. *Suppose that E has property (P). Let $W \subset C_0(X; E)$ be a closed subset such that $h_\varepsilon(f, g) \in W$ for all $f, g \in W$ and $\varepsilon > 0$. Then W has the relative Chebyshev center property in $l_\infty(X; E)$ and, a fortiori, in $C_b(X; E)$ and $C_0(X; E)$.*

Proof. Since $C_0(X; E)$ is closed in $l_\infty(X; E)$, the result follows from Theorem 2.2.

COROLLARY 3.15. *If E has property (P) then $C_0(X; E)$ has the relative Chebyshev center property in $l_\infty(X; E)$ and, a fortiori, in $C_b(X; E)$. In particular, $C_0(X; E)$ admits Chebyshev centers.*

Proof. Let $f, g \in C_0(X; E)$ and $\varepsilon > 0$ be given. There exists a compact set $K \subset X$ such that for every $x \notin K$,

$$\|f(x) - g(x)\| < \varepsilon \qquad \text{and} \qquad \|g(x)\| < \varepsilon.$$

By Definition 2.1,

$$\|h_\varepsilon(f, g)(x)\| < \varepsilon$$

for every $x \notin K$. Therefore,

$$h_\varepsilon(f, g) \in C_0(X; E).$$

COROLLARY 3.16. *If E has property (P) then $c_0(E)$ has the relative Chebyshev center property in $l_\infty(E)$. In particular, $c_0(E)$ admits Chebyshev centers.*

Remark. Yost [26, Lemma 2.6] proved that $c_0(E)$ is an M-ideal in $l_\infty(E)$ when E is *any* Banach space, and therefore, $c_0(E)$ is proximinal in $l_\infty(E)$ even without the hypothesis that E has property (P).

COROLLARY 3.17. *The space c_0 has the relative Chebyshev center property in l_∞. In particular, c_0 admits Chebyshev centers.*

4. Compact Linear Mappings

When E and F are normed spaces, we denote by $\mathscr{L}(E; F)$ the vector space of all bounded linear operators T from E into F with the norm

$$\|T\| = \sup\{\|Tx\|; \|x\| \leqslant 1\}.$$

The subspace of $\mathscr{L}(E; F)$ of compact linear operators will be denoted by $\mathscr{K}(E; F)$.

Let X be a locally compact Hausdorff space and $(E, \| \ \|)$ be a Banach space with dual E^*. We denote by $C_0(X; (E^*, \tau))$ the space of all τ-continuous E^*-valued functions on X vanishing at infinity when E^* has a topology τ. Let

$$C_\sigma(X; E^*) = l_\infty(X; E) \cap C_0(X; (E^*, \sigma)),$$

where $\sigma = \sigma(E^*, E)$ denotes the weak * topology on E^*.

Theorem 4.1. *The space $\mathscr{L}(E, C_0(X))$ is isometrically isomorphic to $C_\sigma(X; E^*)$ via the mapping Φ defined by*

$$\Phi(T)\,x = \delta_x \circ T$$

for all $T \in \mathscr{L}(E; C_0(X))$ and $x \in X$, where δ_x denotes the evaluation map at x. Under this mapping, $\mathscr{K}(E; C_0(X))$ is isometrically isomorphic to $C_0(X; E^*)$.

Remark. This result is well known when X is compact (see Dunford–Schwartz [9, p. 490]). Since the proof in the case of a locally compact Hausdorff space X is a straightforward generalization of the proof in the case of a compact space, we shall omit the proof of Theorem 4.1.

Theorem 4.2. *Let E be a uniformly smooth Banach space. Then $\mathscr{K}(E; C_0(X))$ has the relative Chebyshev center property in $\mathscr{L}(E; C_0(X))$.*

Proof. Since E^* is uniformly convex (see Diestel [7]) by Corollary 3.15, $C_0(X; E^*)$ has the relative Chebyshev center property in $l_\infty(X; E^*)$ and, a fortiori, in $C_\sigma(X; E^*)$. Now the result follows from Theorem 4.1.

Corollary 4.3. *If E is a Hilbert space, or a space l_p or L_p, $1 < p < \infty$ then $\mathscr{K}(E; C_0(X))$ has the relative Chebyshev center property in $\mathscr{L}(E; C_0(X))$.*

Proof. A Hilbert space, or a space l_p or L_p with $1 < p < \infty$ is uniformly smooth (see Clarkson [6]).

Remark. Mach ([16, Corollary 3]) proved that $\mathcal{K}(E; C(X))$ is proximinal in $\mathcal{L}(E; C(X))$ when X is compact and E is a Hilbert space, a space l_p, $1 < p < \infty$, or the space c_0.

COROLLARY 4.4. *If E is a uniformly smooth Banach space then $\mathcal{K}(E; c_0)$ has the relative Chebyshev center property in $\mathcal{L}(E; c_0)$.*

Remark. Mach and Ward [20, Theorem 3.1] and Yost [26, Corollary 2.7] proved that, for any Banach space E, $\mathcal{K}(E; c_0)$ is an M-ideal of $\mathcal{L}(E; c_0)$.

COROLLARY 4.5. *For each $1 < p < \infty$ the space $\mathcal{K}(l_p; c_0)$ has the relative Chebyshev center property in $\mathcal{L}(l_p; c_0)$.*

COROLLARY 4.6. *Let X be a completely regular space and $(E, \|\ \|)$ be a uniformly smooth Banach space. Then $\mathcal{K}(E; C_b(X; \mathbb{K}))$ has the relative Chebyshev center property in $\mathcal{L}(E; C_b(X; \mathbb{K}))$.*

Proof. $\mathcal{L}(E; C_b(X; \mathbb{K}))$ is isometrically isomorphic to $\mathcal{L}(E; C(\beta X; \mathbb{K}))$, where βX denotes the Stone–Čech compactification of X. Under this isomorphism $\mathcal{K}(E; C_b(X; \mathbb{K}))$ can be identified with $\mathcal{K}(E; C(\beta X; \mathbb{K}))$. Now the result follows from Theorem 4.2.

Remark. If we take $\mathbb{K} = \mathbb{R}$ in Corollary 4.6 then the result is true for any topological space X, since $C_b(X; \mathbb{R})$ is an AM-space with unit and so it is isometrically isomorphic to a $C(Y; \mathbb{R})$ for some compact Hausdorff space Y.

Lau ([15, Theorem 4.5(ii)]) proved that for any topological space X, $\mathcal{K}(X; C_b(X; \mathbb{R}))$ is proximinal in $\mathcal{L}(E; C_b(X; \mathbb{R}))$ if E is a uniformly smooth Banach space.

COROLLARY 4.7. *If $(E, \|\ \|)$ is a uniformly smooth Banach space then $\mathcal{K}(E; l_\infty)$ has the relative Chebyshev center property in $\mathcal{L}(E; l_\infty)$.*

Proof. If $\mathbb{N}$ has the discrete topology then $l_\infty = C_b(\mathbb{N}; \mathbb{K})$.

COROLLARY 4.8. *$\mathcal{K}(l_p; l_\infty)$ has the relative Chebyshev center property in $\mathcal{L}(l_p; l_\infty)$ when $1 < p < \infty$.*

Remark. Feder [10, Theorem 1] proved that $\mathcal{K}(l_\infty; l_\infty)$ is *not* proximinal in $\mathcal{L}(l_\infty; l_\infty)$.

Let (S, Σ, μ) be a σ-finite positive measure space and $(F, \|\ \|)$ be a uniformly convex Banach space. We denote by $L_\infty(S, \Sigma, \mu; F)$ the space of all essentially bounded μ-Bochner integrable functions $f\colon S \to F$ normed by

$$\|f\| = \operatorname*{ess\,sup}_{s \in S} \|f(s)\|.$$

The subspace of all elements of $L_\infty(S, \Sigma, \mu; F)$ whose ranges are μ-essentially relatively compact is denoted by $K_\infty(S, \Sigma, \mu; F)$. For definitions see Diestel–Uhl [8].

THEOREM 4.9. *Let $W \subset L_\infty(S, \Sigma, \mu; F)$ be a closed subset such that $h_\varepsilon(f, g) \in W$ for all $f, g \in W$ and $\varepsilon > 0$. Then W has the relative Chebyshev center property in $L_\infty(S, \Sigma, \mu; F)$.*

Proof. We can follow the same reasoning as in Theorem 2.2. The sup–norm is replaced by

$$\|f\| = \operatorname*{ess\,sup}_{s \in S} \|f(s)\|$$

for all $f \in L_\infty(S, \Sigma, \mu; F)$.

THEOREM 4.10. *$\mathscr{K}(L_1(S, \Sigma, \mu); F)$ has the relative Chebyshev center property in $\mathscr{L}(L_1(S, \Sigma, \mu); F)$.*

Proof. Since F is uniformly convex, it has the Radon–Nikodym property (see Clarkson [6]), and so $\mathscr{L}(L_1(S, \Sigma, \mu); F)$ is isometrically isomorphic to $L_\infty(S, \Sigma, \mu; F)$. Also $\mathscr{K}(L_1(S, \Sigma, \mu); F)$ can be identified with $K_\infty(S, \Sigma, \mu; F)$. Let $f, g \in K_\infty(S, \Sigma, \mu; F)$ and $\varepsilon > 0$ be given. As in the proof of Corollary 2.3 we can write

$$h_\varepsilon(f, g)(s) = f(s) + \beta_\varepsilon(\|f(s) - g(s)\|)[g(s) - f(s)]$$

for all $s \in S$. If N_f and N_g denote the μ-null sets such that $K_f = f(S - N_f)$ and $K_g = g(S - N_g)$ are relatively compact, then $N = N_f - N_g$ is a μ-null set and

$$h_\varepsilon(f, g)(s) \in K_f + [0, 1](K_g - K_f)$$

for all $s \in S \backslash N$. Hence, $h_\varepsilon(f, g)(S \backslash N)$ is relatively compact.

Remark. Lau [15, Theorem 4.5(i)] proved the proximinality of $\mathscr{K}(L_1(S, \Sigma, \mu); F)$ under the same hypothesis of Theorem 4.10.

REFERENCES

1. D. AMIR, Chebyshev centers and uniform convexity, *Pacific J. Math.* **77** (1978), 1–6.
2. D. AMIR AND F. DEUTSCH, Approximation by certain subspaces in the Banach space of continuous vector-valued functions, *J. Approx. Theory* **27** (1979), 254–270.
3. G. BACHMAN AND A. SULTAN, Applications of functional analysis to topological measure theory, *in* "Operator Theory and Functional Analysis" (I. Erdelyi, Ed.), pp. 122–164, Research Notes in Mathematics No. 38, Pitman, San Francisco/London/Melbourne, 1979.

4. J. BLATTER, Grothendieck spaces in approximation theory, *Mem. Amer. Math. Soc.* **120** (1972).
5. J. BLATTER AND G. L. SEEVER, Interposition of semicontinuous functions by continuous functions, *in* "Analyse Fonctionnelle et Applications" (L. Nachbin, Ed.), Hermann, Paris, 1975.
6. J. A. CLARKSON, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), 296–414.
7. J. DIESTEL, "Geometry of Banach Spaces—Selected Topics," Lecture Notes in Mathematics No. 485, Springer-Verlag, 1975.
8. J. DIESTEL AND J. J. UHL, JR., "Vector Measures," Mathematical Surveys No. 15, Amer. Math. Soc., Providence, R.I., 1977.
9. N. DUNFORD AND J. SCHWARTZ, "Linear Operators I," Interscience, New York, 1958.
10. M. FEDER, On a certain subset of $L_1(0, 1)$ and non-existence of best approximation in some spaces of operators, *J. Approx. Theory* **29** (1980), 170–177.
11. C. FRANCHETTI AND E. W. CHENEY, Simultaneous approximation and restricted Chebyshev centers in function spaces, *in* "Approximation Theory and Applications" (Z. Ziegler, Ed.), Academic Press, New York, 1981.
12. A. L. GARKAVI, The best possible net and the best possible cross-section of a set in normed space, *Amer. Math. Soc. Transl.* **39** (1964), 111–132.
13. I. M. KADETS AND V. ZAMYATIN, Chebyshev centers in the space $C[a, b]$, *Teor. Funkcii. Funkcional. Anal. i Priložen* **7** (1968), 20–26.
14. E. P. LANE, Insertion of a continuous functions, *Pacific J. Math.* **66** (1976), 181–190.
15. K. S. LAU, Approximation by continuous vector valued functions, *Studia Math.* **68** (1979), 291–299.
16. J. MACH, On the proximinality of compact operators with range in $C(S)$, *Proc. Amer. Math. Soc.* **72** (1978), 99–104.
17. J. MACH, Best simultaneous approximation of bounded functions with values in certain Banach spaces, *Math. Ann.* **240** (1979), 157–164.
18. J. MACH, On the existence of best simultaneous approximation, *J. Approx. Theory* **25** (1979), 258–265.
19. J. MACH, On the proximinality of Stone–Weierstrass subspaces, *Pacific J. Math.* **99** (1982), 97–104.
20. J. MACH AND J. D. WARD, Approximation by compact operators on certain Banach spaces, *J. Approx. Theory* **23** (1978), 274–286.
21. C. OLECH, Approximation of set-valued functions by continuous functions, *Colloq. Math.* **19** (1968), 285–293.
22. J. B. PROLLA, "Approximation of vector valued functions," North-Holland, Amsterdam, 1977.
23. H. H. STONE, Boundedness properties in function lattice, *Canad. J. Math.* **1** (1946), 176–186.
24. P. W. SMITH AND J. D. WARD, Restricted centers in subalgebras of $C(X)$, *J. Approx. Theory* **15** (1975), 54–59.
25. J. D. WARD, Chebyshev centers in spaces of continuous functions, *Pacific J. Math.* **52** (1974), 283–287.
26. D. T. YOST, Best approximation and intersections of balls in Banach spaces, *Bull. Austral. Math. Soc.* **20** (1979), 285–300.